Abstract

Topological states of matter have arisen as one of the rapidly growing fields in condensed matter physics, holding promise for future technologies, including spintronics and quantum computing. The notion of topology underlies exotic phenomena such as emergent collective excitations or protected edge modes exhibited in the low-energy spectrum. In particular, in non-interacting fermionic systems the connection between bulk and boundary degrees of freedom is established by the celebrated *bulk-boundary correspondence*. In the presence (or absence) of fundamental symmetries: time-reversal, particle-hole and chiral, it is possible to classify free-fermionic, gapped systems in different spatial dimensions by the means of the ten-fold way. This classification scheme of topological insulators and superconductors tabulates all possible combinations of aforementioned symmetries and assigns a relevant topological invariant to each symmetry class. Despite its immense success, the ten-fold way turned out to be incomplete in the light of recent theoretical developments and experimental efforts.

This thesis discusses topological aspects of selected free-fermionic systems in low dimensions ($d \leq 2$) that go *beyond* the ten-fold way. We address three distinct research directions in which the existing classification can be extended. In the first part, we investigate the Hofstadter model on two fractal geometries, namely the Sierpiński carpet and gasket. While being embedded in two-dimensional space, these fractals are characterized by a non-integer Hausdorff dimension. In addition, their connectivity properties are in a stark contrast to regular lattices as there is no clear distinction between edges and bulk. We numerically study the spectral and eigenstates localization properties, and observe a hierarchy of edge-like states located at different fractal depths. We employ topological invariants defined in real space: the Bott index and the Chern number, and identify regions in the energy spectrum with non-trivial topology. We further compute the phase diagram in the presence of disorder and conclude that characteristic features of the integer quantum Hall effects are also observed in *almost* two-dimensional systems.

The second part of this dissertation is devoted to the significance of spatial symmetries. We start with concrete examples of group-V monolayers: atomically thin layers of bismuth and antimony, described within a tight-binding approximation. We show that a free-standing layer of bismuth hosts a quantum spin Hall phase, whereas a single layer of antimony has a trivial $\mathbb{Z}_2$ invariant. Applying a moderate strain to free-standing buckled layers, however, results in completely flat structures called bismuthene and antimonene, which are realizing a topological crystalline insulating phase protected by the mirror symmetry along the z axis. Apart from the direct computations of relevant topological invariants, we use entanglement measures as complementary tools to define the bulk topology. We present how the full spectrum of the reduced density matrix corresponding to a spatially separated subsystem allows to differentiate between distinct topological states. Additionally, we study the scaling of the entanglement entropy across different topological phase transitions driven by doping, external electric field and strain. An even more profound consequence of the crystal symmetries is the existence of *obstructed atomic limits*, i.e., systems for which the strong topological indices are trivial, but are not adiabatically connected to a trivial atomic limit. We propose a classification scheme for obstructed atomic limits in two dimensions, where Wilson loops and symmetry indicators play the role of topological invariants. We find that a buckled monolayer of antimony, among other suggested material candidates, is actually an obstructed limit and exhibits symmetry-protected corner charges.

In the third part, we discuss the interplay between topology and non-Hermiticity in Hamiltonians providing an effective description of open systems. Introducing non-Hermiticity leads to unique features such as exceptional points or the anomalous localization of all eigenstates at the boundary (the so-called skin effect). Using the π-flux model on a square lattice with a non-Hermitian extension, we demonstrate a novel phenomenon dubbed the *reciprocal* skin effect, which does not require any direction-dependent hoppings. Theoretical predictions are supported by experimental results obtained by measuring a topolectrical circuit, which realizes the desirable physics of the non-Hermitian π-flux model.

Keywords: topological phases, tight-binding models, fractals, the Hofstadter model, entanglement, topological (crystalline) insulators, non-Hermitian Hamiltonians