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Synopsis of the PhD thesis

Subsets of Polish groups and spaces related to algebraic structure and measurability with respect to variety of ideals

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In this thesis we consider some special subsets of Polish spaces. We will study their properties related to the underlying algebraic and topological structure and measurability with respect to a given ideal. We will mainly focus on $\mathcal{I}$-Luzin sets, tree ideals and we will search for universal sets of possibly low complexity for various ideals. Also, we will revive some classical result concerning a decomposition of the real line into full, with respect to measure or category, subsets.

In chapter 1 we will fix the notation and we will gather some useful facts and notions and we will give context for our considerations. In particular for the purpose of this thesis we will define the technique of fusion for perfect sets, Miller trees and Laver trees. Also, we will identify bodies of certain types of trees on the real line. Furthermore, we will introduce the notion of universal sets for ideals, we will justify such a definition, and we will give a motivation why should we consider such sets in the first place. On top of that we make some nontrivial observations about tree ideals.

The main focus of chapter 2 are $\mathcal{I}$-Luzin sets in Euclidean spaces $\mathbb{R}^n$. First, we will prove a lemma concerning translations of perfects sets. We will use that lemma to conclude that the Weaker Smital Property implies that $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable. Combining it with a result that says that the existence of an $\mathcal{I}$-Luzin set implies the existence of an $\mathcal{I}$-Luzin set whose cardinality has an uncountable cofinality, we will obtain a method of generating super $\mathcal{I}$-Luzin sets from $\mathcal{I}$-Luzin sets. Also, we will provide a condition which is equivalent to measurablity of $\mathcal{I}$-Luzin sets and analyze relations between some significant in that regard properties of ideals. In the next part we will decompose Euclidean spaces $\mathbb{R}^n$ into $\mathcal{I}$-Luzin sets, starting with minimal set of assumptions (there exists an $\mathcal{I}$-Luzin set) and eventually decomposing $\mathbb{R}^n$ into translations of the $\mathcal{I}$-Luzin set by vectors from a set which is an $\mathcal{I}$-Luzin set too. Then we will proceed with various constructions that rely on the linear structure of $\mathbb{R}^n$, usually under some assumptions on cardinal coefficients of the ideal $\mathcal{I}$. One of these results says, under the assumption of existence of an $\mathcal{I}$-Luzin set, that there exists a subset of the plane which intersects each line on an $\mathcal{I}$-Luzin set and which is completely nonmeasurable with respect to some ideal containing lines. We will close the chapter with constructions of a generalized Luzin set L and a Sierpiński set S for which $L+L$ and $S+S$ are Bernstein sets. Most of the results obtained in this chapter are published in [6] and [4].

Chapter 3 is dedicated to tree ideals. In the first part of the chapter we will deal with measurability with respect to tree ideals. Using some notions and techniques introduced by J. Brendle we will enrich his results with our considerations on cl_0 ideal. Namely, we will show that for $t_0 \in \{s_0, m_0, l_0\}$ we have $cl_0 \not\subseteq t_0$, $s_0 \not\subseteq cl_0$ and $m_0 \not\subseteq cl_0$, but $l_0 \subseteq cl_0$. Next we will explore the notion of $\mathbb{T}$-Bernstein sets, in particular we will provide a characterization of these sets by their trace on $t_0 \cap \mathcal{B}$. Then we will focus on algebraic properties connected with families of perfect, Miller, Laver and complete Laver trees. It is realized in the series of lemmas, which we further apply to the main theorem of this chapter, which says that if $\mathfrak{c}$ is a regular cardinal, then the algebraic sum of a generalized

Luzin set and a generalized Sierpiński set belongs to each of the tree ideals s_0, m_0, l_0 and cl_0. The last result for the case of s_0 was published in [6] and its generalization for other tree ideals (with above mentioned lemmas that lead to it) and the considerations on measurability of tree ideals are contained in the article [7], which is sent to review.
In chapter 4 we will aim for constructing or demonstrating the existence of universal sets of possibly low complexity for various ideals possesing Borel base. We will provide a general method of obtaining such sets for $\mathcal{B}$-on-Σ^1_1 ideals and we will give combinatorial proofs in some specific cases. Also, we will show that there exist universal sets for ideals from some class of product ideals, in particular for product ideals $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$. The article, which contains these results, is accepted for print ([2]).
Chapter 5 is a result of a struggle with the Luzin-Novikov theorem. Achieving only partial success in an attempt of understainding the reasoning of Luzin from [3] we will prove the theorem in a different way, obtaining in the process an interesting result on the Fubini Property of the pair $(\mathcal{M}, \mathrm{NWD})$ without the requirement of measurability. Clean and precise proof of the Luzin-Novikov theorem makes up the content of this chapter. The effects of this endeavor are published in [5].

Bibliography

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